$$3\sqrt[3]{26} \quad = \quad 3(26)^3 \quad = \quad 3(26)^{/3}$$

$$\sqrt{a+m} \quad = \quad (a+m)^{\frac{1}{2}} \quad = \quad (a+m)^{/}$$

$$4\sqrt{\frac{a^2+b^2+c^2}{2abc}} \quad = \quad \left(\frac{a^2+b^2+c^2}{2abc}\right)^{\frac{1}{4}} \quad = \quad \left(\frac{a^2+b^2+c^2}{2abc}\right)^{/4}$$

$$\sqrt[5]{(x^3+3xy^2)^3} \quad = \quad (x^3+3xy^2)^{\frac{3}{5}} \text{ , or } (x^3+3xy^2)^{3/5}$$

The proposed notation would do away with vinculums and would use preferably the solidus sign for division as is the tendency now in English mathematical and scientific books. In printing, $\sqrt{\phantom{x}}^{/}$ would be replaced by $)^{/}$ on one type, and in script the latter would be made, without lifting the pen, in loop form. However, when the numerator of the fractional exponent is other than unity, the usual fractional exponent notation (which for this case is preferable to the radical sign notation) would be employed. Notice that by the simple changes proposed, which are perfectly natural ones, all the advantages of the duplicate notation would be preserved with none of its disadvantages, such as the use of the unsightly hieroglyphic-like radical sign (giving as it does a forbidding appearance to the printed page), and the confusion which arises from the simultaneous use of two distinct notations for the same operation.

In conclusion it should be emphasized that mathematicians themselves are not likely to feel the need or approve of any change in the algebraic notation. Like the reform in spelling, it is in the interest chiefly of the hundreds of thousands of students of elementary mathematics yet to come, and not in that of those who have already mastered the two notations, that this reform is urged. Surely it is not too much to ask that the fractional exponents as now written be employed exclusively (instead of largely as now) in all higher works involving the use of algebraical symbols. The abridgments would then be likely to come as a matter of course.

*Stevens Point, Wisconsin, May 11, 1895.*

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# INTRODUCTION TO SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Leipzig, Germany.

[Continued from December Number.]

### THE CONSTRUCTION OF NON-PRIMITIVE GROUPS WITH THREE SYSTEMS OF NON-PRIMITIVITY.

Let the degree of the required group $G$ be $3n$. $G$ must be a subgroup (using subgroup in its broad sense in which it includes the group itself and identity) of

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$$(a_1 a_2 \ldots \ldots a_n)\text{all}(b_1 b_2 \ldots \ldots b_n)\text{all}(c_1 c_2 \ldots \ldots c_n)\text{all}.$$

If $G_1$ is not identity,* its constituents must be conjugate transitive subgroups of these three systems.

If we designate the systems by $A$, $B$, and $C$, the permutations of the systems must correspond to a group of these three letters, for if these permutations would not form a group of operations $G$ itself could not be a group. Hence every non-primitive group with three systems must correspond to one of the following groups:

$$(ABC)\text{cyc} \qquad (ABC)\text{all}$$

Since the former of these is a subgroup of the latter it follows that at least a part of every non-primitive group in three systems corresponds to

$$(ABC)\text{cyc}$$

we proceed to find this part. By a course of reasoning similar to that employed under two systems it follows that all the substitutions which transform any $G_1$ according to $ABC$ must be contained in

$$(a_1 a_2 \ldots \ldots a_n)\text{all}(b_1 b_2 \ldots \ldots b_n)\text{all}(c_1 c_2 \ldots \ldots c_n)\text{all}\ a_1 b_1 c_1 . a_2 b_2 c_2 \ldots \ldots a_n b_n c_n$$

and all those which transform $G_1$ according to $ABC$ must be contained in

$$(a_1 a_2 \ldots \ldots a_n)\text{all}(b_1 b_2 \ldots \ldots b_n)\text{all}(c_1 c_2 \ldots \ldots c_n)\text{all}\ a_1 c_1 b_1 . a_2 c_2 b_2 \ldots \ldots a_n c_n b_n$$

These sets are not independent, for if

$$s_\gamma \qquad \gamma = 1, 2, \ldots (n!)^3$$

represents the substitution of one set then will the $(n!)^3$ different corresponding values of

$$s_\gamma^{-1}$$

represent the substitutions of the other set.

If in any non-primitive group $G_2$ stands for the substitution belonging to the first set and $G_3$ for those belonging to the second set, and if $g_2$ and $g_3$ represent the number of substitutions in $G_2$ and $G_3$ respectively we derive from the fact that if a group contains $s_\gamma$ it must also contain $s_\gamma^{-1}$ that

$$g_2 = g_3$$

If in any non-primitive group we multiply any substitution of $G_2$ by all

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*This case was not considered under two systems of non-primitivity. It was unnecessary to consider it. For, since a transitive group contains substitutions which replace a given letter by all of the letters involved it follows that the order of a non-primitive group is always equal to its degree. It can easily be shown that the order of any transitive group is a multiple of its degree.

the substitutions of $G_3$ we obtain $g_3$ different substitutions of $G_1$, hence

$$g_1 \gtreqless g_3.$$

If we multiply a given substitution of $G_3$ into all the substitutions of $G_1$ we obtain $g_1$ different substitutions of $G_3$, hence

$$g_3 \gtreqless g_1.$$

Combining the last two relations with the preceding we obtain for any non-primitive group with three systems of non-primitivity

$$g_1 = g_2 = g_3.$$

Since the relation between $G_2$ and $G_3$ is such that we can derive one directly from the other we shall generally consider only $G_2$. But $G_2$ can be directly obtained from $G_1$ provided we have given one of the substitutions of $G_2$. Hence to construct the non-primitive group (or the part of a non-primitive group) corresponding to

$$(ABC)\text{cyc}$$

it is only necessary to find $G_1$ and one substitution $(s_\gamma)$ corresponding to $ABC$.

$s_\gamma$ must clearly satisfy the following conditions :

(1) Its cube is found in $G_1$.

(2) It transforms $G_1$ into itself.

(3) It permutes the systems according to $ABC$.

These three conditions are sufficient for if any substitution $s_\gamma$ fulfills these conditions then is

$$G_1 + G_1 s_\gamma + G_1 s_\gamma^{-1}$$

a non-primitive group for

$$G_1 s_{\gamma_1} G_1 = G_1 s_{\gamma_1} G_1 s_\gamma^{-1} s_{\gamma_1} = G_1 s_{\gamma_1}$$

$$G_1 s_\gamma^{-1} G_1 = G_1 s_\gamma^{-1} G_1 s_{\gamma_1} s_\gamma^{-1} = G_1 s_\gamma^{-1}$$

etc., etc., etc.

It remains to prove that the three given conditions are necessary as well as sufficient, i. e., we have to show that none of the three pair of conditions is sufficient. The pair which excludes the last condition is evidently insufficient, and the following examples prove that the other two pair are also insufficient.

| 1 | 1 | 1 | 1 |
|---|---|---|---|
| $abc$ | $def$ | $ghi$ | $abc.def.ghi$ |
| $acb$ | $dfe$ | $gih$ | $acb.dfe.gih$ |
| | | | $ab.de.gh$ |
| | | | $ac.df.gi$ |
| | | | $bc.ef.hi$ |

For *aehbdg.cfi* satisfies the second and third but not the first of the three conditions if we take the first of these groups for $G_1$, and *aehbficdg* satisfies the first and third but not the second if we take the second of these groups for $G_1$. Hence we see that the three given conditions are necessary as well as sufficient.

If the transitive constituents of $G_1$ admit only a cyclical (not a symmetric) permutation then it is impossible to construct a $G$ corresponding to $(ABC)$all and involving the given $G_1$. If they admit a symmetric permutation we have to add to the part of $G$ corresponding to $(ABC)$cyc sufficient substitutions to make it correspond to $(ABC)$all. By a course of reasoning similar to that which we have just pursued we prove that it is only necessary to find one substitution $s_\beta$ corresponding to $AB$, and that $s_\beta$ must satisfy the following conditions :

(1) Interchange the first two systems.

(2) Have its square in $G_1$.

(3) Transform the group corresponding to $ABC$ into itself.

To fix these ideas we proceed to the construction of the non-primitive groups of degree six which contain three systems of non-primitivity. We shall then have found all the non-primitive groups up to degree eight as no such groups can exist for degree seven, or any other prime degree.

### NON-PRIMITIVE GROUPS OF DEGREE SIX WITH THREE SYSTEMS OF NON-PRIMITIVITY.

$G_1$ must be one of the following four groups : $(ab)(cd)(ef)$, $\{ (ab)(cd)$ $(ef) \}$pos, $(ab.cd.ef)$, 1 $G_2$ must be contained in

$$(ab)(cd)(ef)\ ace.bdf$$

(*a*) If $G_1 = (ab)(cd)(ef)$ then will *ace.bdf* evidently satisfy the three necessary conditions, we thus obtain a non-primitive group corresponding to $ABC$, whose order is 24, viz :

(1) $\qquad (ab)(cd)(ef)\ (ace.bdf)\text{cyc} = (abcdef)_{2\,4\,5}$*

For $s_\beta$ we may take *ac.bd.* This leads to a group of order 48 which has the preceding group as a self-conjugate sub-group. The group is

(2) $\qquad (ab)(cd)(ef)(ace.bdf)\text{cyc}(ac.bd) = (abcdef)_{4\,8}$

(*b*) If $G_1 = \{ (ab)(cd)(ef) \}$pos we can again use *ace.bdf* for $s_\gamma$. We thus obtain a second non-primitive group of order 12, viz :

(3) $\qquad \{ (ab)(cd)(ef) \}$pos $(ace.bdf) = (abcdef)_{1\,2\,}$†

This is the only group that corresponds to $ABC$ since the negative substitutions which correspond to the most general $G_2$ do not have their cubes in this

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*The foot note in regard to $(abcdef)_{2\,4_2}$ applies also to this group.

†The foot note in regard to $(abcdef)_{3\,6_3}$ applies also to this group.

$G_1$. For $s_\beta$ we may take both $ac.bd$ and $adbc$. We thus obtain two additional groups of order 24, viz :

(4) $\qquad \Big\langle\, (ab)(cd)(ef)\, \Big\rangle \text{pos}\, (ace.bdf)(ac.bd) = (+abcd)_{24}$

(5) $\qquad \Big\langle\, (ab)(cd)(ef)\, \Big\langle \text{pos}\, (ace.bdf)(adbc) = (\pm abcd)_{24}$

(c) If $G_1 = (ab.cd.ef)$, $s_\gamma$ may again equal $ace.bdf$. The two substitutions $ab.cd.ef$ and $ace.bdf$ generate the group. The first interchanges the two cycles of the second and the second interchanges the three cycles of the first. The resulting group must therefore have two as well as three systems of non-primitivity, and hence is found in the former list. All the other three possible groups corresponding to $ABC$ are conjugate to this.

For $s_\beta$ we may use $ac.bd$, but with $ace.bdf$ this will generate $(ace.bdf)$all. Hence this group is also found in the list of non-primitive groups with two systems of non-primitivity. Hence there is no additional non-primitive group for $G_1 = (ab.cd.ef)$.

(d) If $G_1 = 1$ the second condition of $s_\gamma$ is satisfied by every substitution. The substitutions that may correspond to $ABC$ must be of the third order and are therefore all conjugate so that we need to consider only one of them. We thus obtain the intransitive group

$$(ace.bdf)\text{cyc}.$$

If we take $ac.bd$ for $s_\beta$ we obtain an intransitive group corresponding to $(ABC)$all. If we take $ab.dc.ef$ for $s_\beta$ we obtain a non-primitive group which is also non-primitive in two systems as is evident. Hence $G_1 = 1$ leads to no new non-primitive group.

We have now examined the entire region through degree six with a view to its non-primitive groups and have found the following

### List of Non-Primitive Groups Through Degree Six.

| Degree | Order | No. | Group |
|--------|-------|-----|-------|
| 4 | 4 | 1 | $(abcd)_4$ |
| | | 2 | $(abcd)$cyc |
| | 8 | 1 | $(abcd)_8$ |
| 6 | 6 | 1 | $(abcdef)_6$ |
| | | 2 | $(abcdef)$cyc |
| | 12 | 1 | $(abcdef)_{12}$ |
| | | 2 | $(abcdef)_{12_2}$ |
| | 18 | 1 | $(abcdef)_{18}$ |
| | 24 | 1 | $(+abcdef)_{24}$ |
| | | 2 | $(\pm abcdef)_{24}$ |
| | | 3 | $(abcdef)_{24_5}$ |
| | 36 | 1 | $(abcdef)_{36}$ |

|    | 2 | $(abcdef)_{363}$ |
|----|---|------------------|
| 48 | 1 | $(abcdef)_{48}$  |
| 72 | 1 | $(abcdef)_{72}$  |

### GENERAL REMARKS ON THE CONSTRUCTION OF NON-PRIMITIVE GROUPS.

Let it be required to find the non-primitive groups of degree $n$, $n$ being a composite positive integer greater than three, and let

$$m_1, m_2, \quad . \quad . \quad . \quad m_e$$

be all the positive integral factors of $n$ (excepting unity) which satisfy the relation

$$m\alpha \underset{<}{=} \sqrt[]{n} \quad \begin{matrix} \alpha = 1, 2, \quad . \quad . \quad . \quad e \\ \sqrt{\phantom{n}} \text{ indicates only the arithmetic root.} \end{matrix}$$

Hence we may divide $n$ as follows:

| No. of Systems | No. of Letters in Each System |
|:--------------:|:-----------------------------:|
| $m_1$ | $\dfrac{n}{m_1}$ |
| $m_2$ | $\dfrac{n}{m_2}$ |
| . | . |
| . | . |
| . | . |
| $m_e$ | $\dfrac{n}{m_e}$ |
| $\dfrac{n}{m_1}$ | $m_1$ |
| $\dfrac{n}{m_2}$ | $m_2$ |
| . | . |
| . | . |
| . | . |
| $\dfrac{n}{m_e}$ | $m_e$ |

Two of these relations will become identical when $m_\alpha = \sqrt{n}$ for some value of $\alpha$ in the series

$$1, 2, \quad . \quad . \quad . \quad e.$$

Otherwise they will all be different. From these we see that the number of different ways of dividing $n$ into systems is odd or even as $n$ is or is not a perfect square.

The work of finding all the non-primitive groups for any one of these divisions into systems, e. g. the one which contains $m_1$ systems, may be resolved into the following steps:

(1) Contruct the groups (the $G_1$'s) which have conjugate transitive constituent groups from each of these systems and are so constituted that their con-

stituents admit of the permutations of some transitive group of degree $m_1$. The constituent transitive groups are clearly of degree $\frac{n}{m_1}$ unless $G_1 = 1$. The last case does not need consideration when the order of the transitive group of degree $m_1$ is not a multiple of $n$.

[To be Continued.]

# NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from December Number.]

SCHOLION IV : *In which is expounded on a figure a certain consideration on which Euclid probably thought, in order to establish that Postulate of his as 'per se' evident.*

I premise first: within any acute angle $BAX$ (Fig. 12.) can be drawn from any point $X$ of $AX$ a certain straight $XB$, which under designated even if obtuse angle $R$, which only with this acute $BAX$ falls short of two right angles; a certain $XB$, say I, can be drawn, which at a finite remove meets this $AB$ in a certain point $B$. For just that I have demonstrated in a Scholion after P. XIII. I premise secondly: these $AB$, $AX$ (Fig. 25) can be understood as produced into the infinite even to certain points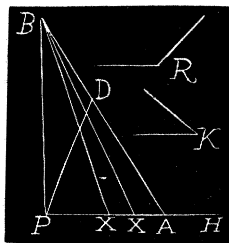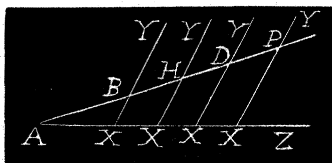 $Y$, and $Z$; and likewise the afore-said $XB$ (into the infinite and itself produced even to a certain point $Y$) can be understood to be so moved above this $AB$ toward the parts of the point $Z$, that the angle at the point $X$ toward the parts of the point $A$ is always equal to the certain given obtuse angle $R$.



Fig. 12.



Fig. 25.

I premise thirdly: that Euclidean Postulate would be liable now to no doubt, if the aforesaid $XY$ in this however great motion above the straight $AZ$ cuts always that $AY$ in certain points $B$, $D$, $H$, $P$, and so successively in other points more remote from this point $A$.

The reason is evident; since thus any two straights $AB$, $XH$ lying in the same plane, upon which any straight $AX$ cutting makes two angles toward the